AMS210.01.
Homework 4

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Due at the beginning of the class, March 28, 2003

1. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that
$$f(x, y, z) = (x + 2z,\ 2x - y - z)$$
and $g : \mathbb{R}^2 \to \mathbb{R}^3$ such that
$$g(x, y) = (2x + y,\ x + 2y,\ x + y).$$
Find $(f \circ g)$ and $(g \circ f)$.

2. Check which of the following functions are linear. Justify your answer.
   (a) $f(x, y) = (x,\ y^2)$
   (b) $f(x, y) = (|x + y|,\ x - y)$
   (c) $f(x, y, z) = (x + y,\ z,\ 0)$
   (d) $f(x, y) = ((x + 1)^2,\ y + 3)$
   (e) $f(x, y, z) = (x + y + z,\ y + z,\ z)$

3. Let $f : M_{n,n} \to \mathbb{R}$ be a function which maps any $n \times n$ matrix to a sum of its diagonal elements. Determine is this function linear or not.

4. Let V be a vector space of all $n \times n$-matrices, and M be a fixed matrix in V. Which of the following functions $T_i : V \to V$ are linear:
   (a) $T_1(X) = XM$
   (b) $T_2(X) = X + M$
   (c) $T_3(X) = XM - MX$
   (d) If M is invertible, $T_4(X) = MXM^{-1}$

5. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that $f(x, y, z) = (2x + y,\ x + y + z)$.
   (a) Check that f is a linear function.
   (b) Find the dimension of the kernel of f and its basis.
   (c) Find the dimension of the image of f and its basis.

(d) Find the matrix of this function in standard basis.

6. Let $f : \mathbb{R}^5 \to \mathbb{R}^3$ such that $f(x, y, z, s, t) = (x+2y-2z+s+3t,\ 2x+y+z-2s-2t,\ 5x+4y-3s-t)$.

   (a) Check that f is a linear function.

   (b) Find the dimension of the kernel of f and its basis.

   (c) Find the dimension of the image of f and its basis.

   (d) Find the matrix of this function in standard basis.

7. Let $f : \mathbb{P}_4 \to \mathbb{P}_4$ such that f is a function of taking a second derivative of a polynomial, i.e., for $p(t) = a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0$ we have:

$$f(p) = D(D(p)) = D(4a_4t^3 + 3a_3t^2 + 2a_2t + a_1) = 12a_4t^2 + 6a_3t + 2a_2,$$

   where D is a function of taking a derivative.

   (a) Compute $f(5t^4 + 2t^3 - 4t + 1)$.

   (b) Find the formula for $f(f(p))$.

   (c) Prove that f is a linear function.

   (d) Find the basis and the dimension of the kernel of f.

   (e) Find the basis and the dimension of the image of f.

   (f) Find the matrix of this linear function.

8. The following matrix

$$A = \begin{pmatrix} 1 & 3 & 2 & -1 \\ 2 & 2 & 0 & 2 \\ 4 & 8 & 4 & 0 \end{pmatrix}$$

   determines a linear function $f_A : \mathbb{R}^4 \to \mathbb{R}^3$.

   (a) Find the basis and the dimension of the kernel of f_A.

   (b) Find the basis and the dimension of the image of f_A.

9. (a) Let V be a vector space, and let $\mathbf{0} : V \to V$ is a zero function, such that $\mathbf{0}(v) = \mathbf{0}$ for any vector v. Find the image and the kernel of $\mathbf{0}$.

   (b) Let V be a vector space, and let $I : V \to V$ is an identity function, such that $I(v) = v$ for any vector v. Find the image and the kernel of I.

10. **[Extra credit]** Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a fixed matrix. Let's consider the linear function $f : \mathbb{P}_1 \to M_{2,2}$ such that $p(t) \mapsto p(A)$. Find the matrix of a f, if a basis in $\mathbb{P}_1$ is $\{t, 1\}$, and in $M_{2,2}$ is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

11. **[Extra credit]** Let $\mathbb{F}_q$ be finite field with q elements. Find the number of linear functions from $\mathbb{F}_q^k$ to $\mathbb{F}_q^n$, where $\mathbb{F}_q^k$ and $\mathbb{F}_q^n$ denote the spaces of all k-tuples and n-tuples respectively of elements from $\mathbb{F}_q$.